Standing waves for a Gauged Nonlinear Schrödinger equation

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Outline

The problem

Consider a planar gauged Nonlinear Schrödinger Equation:

$$iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0.$$

Here $t \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is the scalar field, $A_\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are the components of the gauge potential and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative ($\mu = 0,\ 1,\ 2$).

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In Chern-Simons theory, a modified gauge field equation has been introduced [Hagen, Jackiw, Schonfeld, Templeton, in the '80s]; see also [Tarantello, PNLDE 2007.]

$$\partial_\mu F^{\mu\nu} + \frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Here $\kappa \in \mathbb{R}$ is the Chern-Simons constant and $\epsilon^{\nu\alpha\beta}$ is the Levi-Civita tensor. Moreover, j^μ is the conserved matter current,

$$j^0 = |\phi|^2, \; j^i = 2\mathrm{Im}\left(\bar{\phi}D_i\phi\right).$$

At low energies, the Maxwell term becomes negligible and can be dropped, giving rise to:

$$\frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^{\nu}.$$

See [Jackiw & Pi, '90s].

Taking for simplicity $\kappa = 2$, we arrive to the system

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0, \\ \partial_0 A_1 - \partial_1 A_0 = \mathrm{Im}(\bar{\phi}D_2\phi), \\ \partial_0 A_2 - \partial_2 A_0 = -\mathrm{Im}(\bar{\phi}D_1\phi), \\ \partial_1 A_2 - \partial_2 A_1 = \frac{1}{2}|\phi|^2, \end{cases} \tag{1}$$

As usual in Chern-Simons theory, problem (1) is invariant under gauge transformation,

$$\phi \to \phi e^{i\chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi,$$

for any arbitrary C^∞-function χ.

The initial value problem for $p = 3$, as well as global existence and blow-up, has been addressed in [Bergé, de Bouard & Saut, 1995; Huh, 2009-2013; Liu-Smith-Tataru 2013; Oh-Pusateri, preprint; Liu-Smith, preprint; Chen-Smith, preprint].

The existence of standing waves for (1) and general $p > 1$ has been studied in [Byeon, Huh & Seok, 2012 and preprint]. They look for *vortex solutions*, i.e., solutions in the form:

$$\phi(t,x) = u(r)e^{i(N\theta+\omega t)}, \qquad A_0(x) = A_0(|x|),$$
$$A_1(t,x) = -\frac{x_2}{|x|^2}h(|x|), \qquad A_2(t,x) = \frac{x_1}{|x|^2}h(|x|).$$

Here (r,θ) are polar coordinates, h is a positive function and $N \in \mathbb{N}$ is the order of the vortex at 0.

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With this ansatz they obtain the nonlocal equation:

$$-\Delta u + \left(\omega + \frac{(h_u(|x|) - N)^2}{|x|^2} + A_0(|x|)\right) u = |u|^{p-1}u, \qquad (\mathcal{P})$$

with

$$h_u(r) = \int_0^r \frac{s}{2}u^2(s)\, ds, \;\; A_0(r) = \int_r^{+\infty} \frac{h(s) - N}{s}u^2(s)\, ds.$$

Moreover, any solution satisfies that $u(|x|) \sim |x|^N$ around the origin.

In [Byeon, Huh & Seok, 2012 and preprint] it is shown that $(\mathcal{P})$ is indeed the Euler-Lagrange equation of the energy functional:

$$I_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right)\, dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx$$
$$+ \frac{1}{8}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} u^2(s)s\, ds - 2N\right)^2 dx$$

That functional is defined in the space:

$$\mathcal{H} = \left\{ u \in H^1_r(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} dx < +\infty \right\}.$$

It can be proved that I_ω is well-defined and C^1.

A useful inequality

In [Byeon, Huh & Seok 2012 and preprint], it is proved that, for any $u \in \mathcal{H}$,

$$\int_{\mathbb{R}^2} |u(x)|^4 \, dx$$

$$\leqslant 2\left(\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} u^2(s) s \, ds - 2N\right)^2 dx\right)^{\frac{1}{2}}.$$

Furthermore, the equality is attained by the family of functions:

$$\left\{ u_\lambda = \frac{\sqrt{8}\lambda(N+1)|\lambda x|^N}{1+|\lambda x|^{2(N+1)}} \in \mathcal{H} : \lambda \in (0, +\infty) \right\}.$$

Byeon-Huh-Seok results

- If $p > 3$, I_ω is unbounded from below and exhibits a mountain-pass geometry.
- The case $p = 3$ is special: static solutions can be found via the minimizers of the previous inequality. Alternatively, one can pass, via a self-dual equation, to a singular Liouville equation in $\mathbb{R}^2$.
- If $1 < p < 3$ solutions are found as minimizers on a L^2-sphere if $N = 0$. Hence, ω comes out as a Lagrange multiplier, and it is not controlled.

In general, the global behavior of the energy functional I_ω is not studied for $1 < p < 3$. This is the main purpose of this talk.

On the boundedness from below of I_ω

Theorem

Let $N \in \mathbb{N}$, $p \in (1,3)$. There exists $\omega_0(p) > 0$ such that:

- *If $0 < \omega < \omega_0$, then I_ω is unbounded from below.*
- *If $\omega > \omega_0$, then I_ω is bounded from below and coercive.*
- *If $\omega = \omega_0$, then I_{ω_0} is bounded from below, not coercive and $\inf I_{\omega_0} < 0$.*

The threshold value ω_0 has an explicit expression, and it is independent of N.

A. Pomponio and D. R., 2015.

Y. Jiang, A. Pomponio and D.R., preprint.

The limit functional

Let u a fixed function, and define $u_\rho(r) = u(r-\rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho \to +\infty$.

$$(2\pi)^{-1} I_\omega(u_\rho) = \frac{1}{2}\int_0^{+\infty} (|u_\rho'|^2 + \omega u_\rho^2) r\,dr + \frac{1}{8}\int_0^{+\infty} \frac{u_\rho^2(r)}{r}\left(\int_0^r s u_\rho^2(s)\,ds - 2N\right)^2 dr - \frac{1}{p+1}\int_0^{+\infty} |u_\rho|^{p+1} r\,dr.$$

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\begin{aligned}
(2\pi)^{-1} I_\omega(u_\rho) \sim \ & \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2)(r+\rho)\, dr \\
& + \frac{1}{8} \int_{-\infty}^{+\infty} \frac{u^2(r)}{r+\rho} \left(\int_{-\infty}^{r} (s+\rho) u^2(s)\, ds - 2N \right)^2 dr \\
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& - \frac{1}{p+1}\int_{-\infty}^{+\infty} |u|^{p+1} \rho \, dr.
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$$\begin{aligned}(2\pi)^{-1} I_\omega(u_\rho) \sim \rho \Bigg[& \frac{1}{2}\int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2)\, dr \\ & + \frac{1}{8}\int_{-\infty}^{+\infty} u^2(r) \left(\int_{-\infty}^{r} u^2(s)\, ds \right)^2 dr \\ & - \frac{1}{p+1}\int_{-\infty}^{+\infty} |u|^{p+1}\, dr \Bigg].\end{aligned}$$

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$$(2\pi)^{-1} I_\omega(u_\rho) \sim \rho \left[\frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2)\, dr + \frac{1}{24} \left(\int_{-\infty}^{+\infty} u^2(r) dr \right)^3 - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1}\, dr \right].$$

It is natural to define the limit functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$,

$$J_\omega(u) = \frac{1}{2}\int_{-\infty}^{+\infty} \left(|u'|^2 + \omega u^2\right) dr + \frac{1}{24}\left(\int_{-\infty}^{+\infty} u^2 dr\right)^3 - \frac{1}{p+1}\int_{-\infty}^{+\infty} |u|^{p+1}\, dr.$$

We have

$$I_\omega(u_\rho) \sim 2\pi\rho\, J_\omega(u), \quad \text{as } \rho \to +\infty.$$

Then,

$$\inf J_\omega < 0 \Rightarrow \inf I_\omega = -\infty.$$

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We will actually show that

$$\inf J_\omega < 0 \Leftrightarrow \inf I_\omega = -\infty.$$

The limit functional

Proposition

Let $p \in (1,3)$ and $\omega > 0$. Then J_ω is coercive and attains its infimum.

The proof of the coercivity is based on the Gagliardo-Nirenberg inequality:

$$\|u\|_{L^4(\mathbb{R})} \leqslant C\|u'\|_{L^2(\mathbb{R})}^{1/4}\|u\|_{L^2(\mathbb{R})}^{3/4}.$$

Hence

$$\int_{-\infty}^{+\infty} u^4\,dr \leqslant \frac{C}{2}\left[\int_{-\infty}^{+\infty} |u'|^2\,dr + \left(\int_{-\infty}^{+\infty} u^2\,dr\right)^3\right].$$

The limit problem

The Euler-Lagrange equation of the functional J_ω is:

$$-u'' + \underbrace{\left[\omega + \frac{1}{4}\left(\int_{-\infty}^{+\infty} u^2(s)\,ds\right)^2\right]}_{k} u = |u|^{p-1}u, \quad \text{in } \mathbb{R}. \qquad (2)$$

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Then $u = \pm w_k$ up to translations, where

$$w_k(r) = k^{\frac{1}{p-1}} w_1(\sqrt{k} r),$$

and

$$w_1(r) = \left(\frac{2}{p+1}\cosh^2\left(\frac{p-1}{2} r\right)\right)^{\frac{1}{1-p}}.$$

Therefore,

$$k = \omega + \frac{1}{4}\left(\int_{-\infty}^{+\infty} w_k(r)^2\,dr\right)^2 = \omega + \frac{1}{4}m^2k^{\frac{5-p}{p-1}},$$

where

$$m = \int_{-\infty}^{+\infty} w_1(r)^2\,dr.$$

Proposition

u is a nontrivial solution of (2) if and only if $u(r) = \pm w_k(r - \xi)$ for some $\xi \in \mathbb{R}$ and k is a root of the equation

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Moreover, there exists $\omega_1 > 0$ such that:

- *If $\omega > \omega_1$, (3) has no solution.*
- *If $\omega = \omega_1$, (3) has only one solution k_0.*
- *If $\omega \in (0, \omega_1)$, (3) has two solutions $k_1(\omega) < k_2(\omega)$.*

Moreover,

$$\omega_1 = \left(\frac{(5-p)m^2}{4(p-1)} \right)^{-\frac{p-1}{2(3-p)}} - \frac{m^2}{4} \left(\frac{(5-p)m^2}{4(p-1)} \right)^{-\frac{(5-p)}{2(3-p)}}.$$

The threshold value ω_0

Hence, for $\omega \in (0, \omega_1)$ there are three solutions: 0, w_{k_1} and w_{k_2}. By evaluating J_ω, we obtain that $J_\omega(0) = 0$, $J_\omega(w_{k_1}) > 0$ and

$$J_\omega(w_{k_2}) < 0 \Leftrightarrow \omega < \omega_0,$$

with

$$\omega_0 = \frac{3-p}{3+p}\, 3^{\frac{p-1}{2(3-p)}}\, 2^{\frac{2}{3-p}} \left(\frac{m^2(3+p)}{p-1} \right)^{-\frac{p-1}{2(3-p)}}.$$

Moreover $J_{\omega_0}(w_{k_2}) = 0$.

For some values of p, m can be computed, and hence ω_0. For instance, if $p = 2$, $m = 6$ and $\omega_0 = \frac{2}{5\sqrt{15}}$.

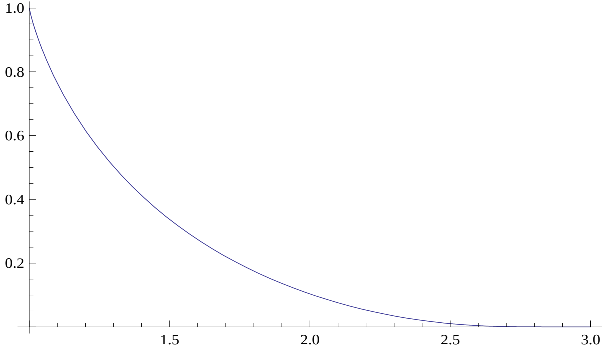

Figura: The value $\omega_0(p)$ for $p \in (1,3)$.

Theorem

Let $p \in (1,3)$. *We have:*

- *if* $\omega \in (0, \omega_0)$, *then* I_ω *is unbounded from below;*
- *if* $\omega = \omega_0$, *then* I_{ω_0} *is bounded from below, not coercive and* $\inf I_{\omega_0} < 0$*;*
- *if* $\omega > \omega_0$, *then* I_ω *is bounded from below and coercive.*

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- *if $\omega > \omega_0$, then I_ω is bounded from below and coercive.*

We estimate $I_\omega(w_{k_2}(\cdot - \rho))$, obtaining:

$$I_\omega(w_{k_2}(\cdot - \rho)) = 2\pi\rho J_\omega(w_{k_2}) - C + o_\rho(1), \text{ as } \rho \to +\infty, \ C > 0.$$

Since $J_\omega(w_{k_2}) < 0$ for $\omega \in (0,\omega_0)$ the first part is proved.

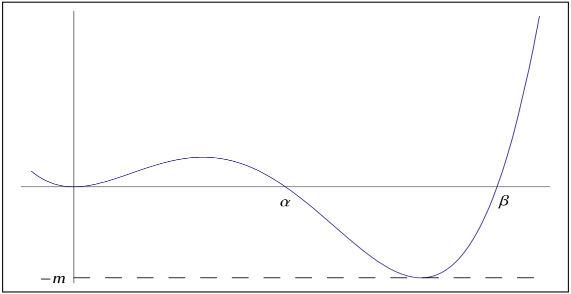
Moreover, $J_{\omega_0}(w_{k_2}) = 0$, so I_{ω_0} is not coercive and $\inf I_{\omega_0} < 0$.

I_ω bounded from below if $\omega \geq \omega_0$.

By using BHS inequality,

$$(2\pi)^{-1} I_\omega(u) \geqslant \frac{1}{4}\|u\|^2 + \frac{1}{16}\int_0^{+\infty} \frac{u^2(r)}{r}\left(\int_0^r s u^2(s)\,ds - 2N\right)^2 dr + \int_0^{+\infty} f(u) r\,dr. \qquad (4)$$

Here $\|\cdot\|$ is the $H_r^1(\mathbb{R}^2)$ norm and $f(u) = \omega\frac{u^2}{2} + \frac{u^4}{4} - \frac{u^{p+1}}{p+1}$.



Define

$$A(u) = \{x \in \mathbb{R}^2 : u(x) \in (\alpha, \beta)\},\ \rho(u) = \sup\{|x| : x \in A(u)\}.$$

Then we obtain:

$$\frac{I_\omega(u)}{2\pi} \geqslant \frac{1}{4}\|u\|^2 + \frac{1}{16}\int_0^{+\infty} \frac{u^2(r)}{r}\left(\int_0^r su^2(s)\,ds - 2N\right)^2 dr - m|A(u)|.$$

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In particular, I_ω is coercive when restricted to $H_0^1(B(0,n))$. Take u_n a minimizer, and observe that

$$I_\omega(u_n) \to \inf I_\omega, \text{ as } n \to +\infty.$$

If u_n is bounded we are done, so let us assume that $\|u_n\|$ diverges. In particular $|A_n|$ must diverge, and hence ρ_n.

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It can be proved that indeed $\rho_n \sim |A_n| \sim \|u_n\|^2$.

By concentration-compactness, we can prove the existence of $\xi_n \sim \rho_n$ such that

$$0 < c < \int_{\xi_n - 1}^{\xi_n + 1} (u_n^2 + u_n')^2 \, dr < C.$$

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Take a cut-off function ψ_n such that

$$\psi_n(r) = \begin{cases} 0, & \text{if } r \leqslant \xi_n - 3\|u_n\|, \\ 1, & \text{if } r \geqslant \xi_n - 2\|u_n\|. \end{cases}$$

We now split the expression of I_ω, but an extra term comes due to its non-local character:

$$
\begin{aligned}
I_\omega(u_n) \geqslant\, & I_\omega(u_n\psi_n) + I_\omega\left(u_n(1-\psi_n)\right) \\
& + c\|u_n(1-\psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|).
\end{aligned}
$$

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I_\omega(u_n) \geqslant{}& 2\pi\xi_n J_\omega(u_n\psi_n) + I_\omega(u_n(1-\psi_n)) \\
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\end{aligned}
$$

Since $\|u_n\psi_n\|_{H^1(\mathbb{R})} \nrightarrow 0$, for $\omega > \omega_0$, we can prove that $J_\omega(u_n\psi_n) \to c > 0$.

Hence, $I_\omega(u_n) > I_\omega(u_n(1-\psi_n))$, which is a contradiction with the definition of u_n.

$$
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Hence, $I_\omega(u_n) > I_\omega(u_n(1-\psi_n))$, which is a contradiction with the definition of u_n.
If $\omega = \omega_0$, we reach a contradiction unless $\psi_n u_n(\cdot - \xi_n) \to w_{k_2}$.
With this extra information, we have a better estimate:

$$
\begin{aligned}
I_{\omega_0}(u_n) \geqslant 2\pi\xi_n J_{\omega_0}(u_n\psi_n) + I_{\omega_0}\big(u_n(1-\psi_n)\big) \\
+ c\|u_n(1-\psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(1).
\end{aligned}
$$

Therefore

$$
I_{\omega_0}(u_n) \geqslant I_{(\omega_0+2c)}\big(u_n(1-\psi_n)\big) + O(1) \geq O(1).
$$

On the solutions of $(\mathcal{P})$

Theorem

- *If ω is large, then $(\mathcal{P})$ has no solutions different from zero.*
- *If $\omega > \omega_0$ is close to ω_0, then $(\mathcal{P})$ admits at least two positive solutions.*
- *For almost every $\omega \in (0, \omega_0)$, $(\mathcal{P})$ admits a positive solution.*

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- *For almost every $\omega \in (0, \omega_0)$, $(\mathcal{P})$ admits a positive solution.*

- **Non-existence of solutions if ω large.**

If $N = 0$, the proof is very simple: multiply the equation by u, integrate and plug the BHS inequality, to get

$$0 \geqslant \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} \left(\omega u^2 + \frac{3}{4} u^4 - |u|^{p+1} \right) dx.$$

And this is a contradiction for ω large.

For $N > 0$ this proof becomes delicate, and will be skipped in this talk.

- **Two solutions if $\omega > \omega_0$ is close to ω_0.**

  Recall that $\inf I_{\omega_0} < 0$, then $\inf I_\omega < 0$ for ω close to ω_0. Being I_ω coercive, the infimum is attained (at negative level).

  Moreover, I_ω satisfies the geometrical assumptions of the Mountain Pass Theorem.

  Since I_ω is coercive, (PS) sequences are bounded.

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  Moreover, I_ω satisfies the geometrical assumptions of the Mountain Pass Theorem.

  Since I_ω is coercive, (PS) sequences are bounded.

  We find a second solution (a mountain-pass solution) which is at a positive energy level.

- **For almost every $\omega \in (0, \omega_0)$ there is a positive solution.**

  If $\omega < \omega_0$, the functional I_ω satisfies the geometric properties of the Mountain-Pass lemma.

  However, (PS) sequences could be unbounded. Here we use the so-called monotonicity trick of Struwe. In this way we can obtain solutions, but only for almost every $\omega \in (0, \omega_0)$.

Thank you for your attention!